UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH3506**
ASSESSMENT PATTERN	:	**MATH3506A**
MODULE NAME	:	**Mathematical Ecology**
DATE	:	**08-May-13**
TIME	:	**14:30**
TIME ALLOWED	:	**2 Hours 0 Minutes**

2012/13-MATH3506A-001-EXAM-31

TURN OVER

All questions may be attempted but only marks obtained on the best four *solutions will count.*
The use of an electronic calculator is not *permitted in this examination.*

1. A fish population of density N_k at generation k grows according to the discrete time model

$$N_{k+1} = \frac{bN_k}{1+N_k^2} = f(N_k), \quad k = 0, 1, 2\ldots, \tag{1}$$

where $b > 1$.

(a) Find all steady state populations.

(b) Determine the linear stability of all steady state populations.

(c) Sketch the cobweb map for (1) when $b > 2$.

The fish population is now harvested, so that the population grows according to

$$N_{k+1} = \frac{bN_k}{1+N_k^2} - HN_k, \quad k = 0, 1, 2\ldots, \tag{2}$$

where $H > 0$ and $b > H + 1$.

(d) Explain what happens to the fish population for $b > b_c = \dfrac{(1+H)^2}{H}$ and sketch the cobweb map to illustrate your answer.

2. The growth of a population of density $N(t)$ at time t is modelled by the differential equation

$$\frac{dN}{dt} = \rho N\left(1 - \frac{N}{K(t)}\right), \quad N(0) = N_0, \tag{3}$$

where $\rho > 0$ and $N_0 > 0$ are constants and $K(t) > 0$ is a periodic function with period T.

(a) By considering $M(t) = N(t)\,e^{-\rho t}$, or otherwise, show that (3) has the solution

$$N(t) = \frac{N_0 e^{\rho t}}{1 + N_0 \int_0^t H(u)\,du}, \qquad \text{where } H(u) = \frac{\rho e^{\rho u}}{K(u)}.$$

(b) Show that for integer $k \geqslant 0$ and $s \in [0, T)$,

$$\int_0^{kT+s} H(u)\,du = \left(\frac{1-e^{k\rho T}}{1-e^{\rho T}}\right)\int_0^T H(u)\,du + e^{k\rho T}\int_0^s H(u)\,du.$$

(c) Find $N_\infty(s) = \lim_{k\to\infty} N(kT+s)$ and show that $N_\infty(0) = N_\infty(T)$.

3. A model for the interaction of two species of densities x_1, x_2 is given by

$$\begin{aligned}\frac{dx_1}{dt} &= r_1 x_1 \left(1 - \frac{x_1}{K_1}\right) - c_1 x_1 x_2 \\ \frac{dx_2}{dt} &= r_2 x_2 \left(1 - \frac{x_2}{K_2}\right) - c_2 x_1 x_2,\end{aligned} \tag{4}$$

where $r_1 > 0, r_2 > 0, K_1, K_2 > 0$ and $c_1, c_2 > 0$.

(a) What kind of interspecies interaction does the system (4) model? What do the K_1, K_2 represent in ecological terms?

(b) Rewrite the system (4) in the form

$$\begin{aligned}\frac{du_1}{d\tau} &= u_1(1 - u_1 - a_{12}u_2) \\ \frac{du_2}{d\tau} &= ru_2(1 - u_2 - a_{21}u_1),\end{aligned} \tag{5}$$

where $\tau = r_1 t$, $u_1 = x_1/K_1$, $u_2 = x_2/K_2$, and r, a_{12}, a_{21} are parameters which you should find.

(c) Determine all possible steady states of (5) and characterise the local stability of any non-zero steady state if it exists.

(d) Sketch the phase plane for (5) in the case $a_{12} < 1, a_{21} < 1$.

4. In a simple model for a population of reptiles, the population is divided into 2 classes: the juvenile class J and the adult class A. Juveniles cannot reproduce, and all adults reproduce at the rate b_A. The probability of a newborn surviving to a juvenile is p_J, the probability of a juvenile surviving to adulthood is p_A, and the probability of an adult surviving thereafter from one time unit to the next is p_S where $p_S < p_A - p_J b_A$.

(a) Derive equations for the size of the juvenile population J_{k+1} and adult population A_{k+1} at generation $k+1$ in terms of generation $k \geqslant 0$ and find the Leslie matrix for the model.

(b) If X_k is the juvenile fraction of the population at generation k (excluding newborns), show that

$$X_{k+1} = \frac{1 - X_k}{\beta(1 - X_k) + \alpha X_k},$$

where α, β are constants which you should find in terms of p_A, p_J, p_S and b_A.

(c) Suppose now that $p_S = 0$. Prove that the fraction of juveniles in the population changes periodically for all $\alpha > 0$. Are non-trivial cycles of length 3 possible?

5. A predator-prey model has the form

$$\begin{aligned}\frac{dX}{dt} &= \rho X\left(1-\frac{X}{K}\right) - \frac{\gamma XY}{A+X} = F(X,Y)\\ \frac{dY}{dt} &= \left(\frac{\sigma XY}{A+X}\right) - \mu Y = G(X,Y),\end{aligned} \tag{6}$$

where $\rho, K, \gamma, A, \sigma$ are all positive constants and

$$\sigma > \mu. \tag{7}$$

(a) Which of X, Y represents the predator, and which the prey? What does the parameter γ represent?

(b) If in addition to (7), $A\mu < (\sigma-\mu)K$, show that (6) has a unique interior steady state (X^*, Y^*) and find X^*, Y^* in terms of the given parameters.

(c) Find the stability matrix $J = \begin{pmatrix} \frac{\partial F}{\partial X} & \frac{\partial F}{\partial Y} \\ \frac{\partial G}{\partial X} & \frac{\partial G}{\partial Y} \end{pmatrix}$ and show that the trace of J at the interior steady state is

$$\frac{\rho X^*}{K(A+X^*)}(K - A - 2X^*).$$

(d) Given that $K > A$, find the critical value of σ/μ above which a stable limit cycle appears.

(e) If the inequality was reversed in (7), what would happen to the predator and prey populations in the long term? Explain your answer carefully.